

Improved Lattice Gauge Field Hamiltonian

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Abstract

Lepage's improvement scheme is a recent major progress in lattice *QCD*, allowing to obtain continuum physics on very coarse lattices. Here we discuss improvement in the Hamiltonian formulation, and we derive an improved Hamiltonian from a lattice Lagrangian free of $O(a^2)$ errors. We do this by the transfer matrix method, but we also show that the alternative via Legendre transformation gives identical results. We consider classical improvement, tadpole improvement and also the structure of Lüscher-Weisz improvement. The resulting color-electric energy is an infinite series, which is expected to be rapidly convergent. For the purpose of practical calculations, we construct a simpler improved Hamiltonian, which includes only nearest-neighbor interactions.

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1 Introduction

Lepage's improvement scheme[1] is a recent major progress in lattice QCD , opening the possibility to approach continuum physics on very coarse lattices. In this paper we want to address the question how this improvement scheme can be formulated for a lattice gauge theory in the Hamiltonian approach. Although standard lattice gauge theory has been very successful over the last two decades, there are areas where progress has been quite slow. Examples are the dynamical computation of the S -matrix and cross sections, QCD at finite baryon density, or the computation of QCD structure functions in the region of small x_B and Q^2 . This situation calls for the development of new methods, and in our opinion the lattice Hamiltonian approach is a viable alternative [2] which should be explored. The Hamiltonian approach corresponds to consider a continuous time, i.e. $a_t = 0$. Similar ideas have been pursued recently by workers in standard lattice gauge theory by considering anisotropic lattices with lattice spacings $a_t \ll a_s$. This has the purpose to improve the computation of the mass spectrum [1, 3].

As a first step, we here restrict ourselves to the problem of improvement in pure gauge theory. Let us recall that the improvement program by Lepage consists of several steps, by starting from the Wilson action: First defining a classically improved action, second performing tadpole improvement and third introducing additional quantum corrections (Lüscher-Weisz improvement). We now discuss how to carry over these ideas to the Hamiltonian formulation. Different strategies are possible, let us explain those for the case of classical improvement.

Strategy 1: Construct the classical Hamiltonian corresponding to the classical Wilson action. Improve this classical Hamiltonian and quantize this Hamiltonian according to the rules of canonical quantization. This yields a *classically improved quantum Hamiltonian*.

Strategy 2: Starting from the classically improved Wilson action, construct via the transfer matrix a classically improved quantum Hamiltonian.

Strategy 3: Starting from the classical Wilson action, construct first the corresponding quantum Hamiltonian via the transfer matrix. This yields the Kogut-Susskind Hamiltonian $H(E, U)$ where U and E are the link variables and their canonical conjugate momenta. The usual expansion in powers of the field variables A and their conjugate $\partial/\partial A$ yields then the standard

expression $\int d^3x((\partial/\partial A)^2 + B^2(A))$ up to correction of (a^2) . This Kogut-Susskind Hamiltonian may be improved by adding corrections such that an better agreement with this *formal* continuum limit operator is obtained.

Different strategies exist also for the further improvements (tadpole and Lüscher-Weisz) with respect to the quantum Hamiltonian. In principle, the construction of these quantum corrections should start from new perturbative calculations in the Hamiltonian framework. The coefficients of a fully improved action as given by Lepage can only be used as starting point for the transfer matrix to obtain a fully improved Hamiltonian if the action is expressed on a lattice with the time spacing being much smaller than the spatial spacing ($a_t \ll a_s$).

In this paper we restrict ourselves mostly to a discussion of classical improvement. As a first result we show explicitly that the first strategy - the canonical quantization of a classical lattice gauge theory - is a viable alternative to the second strategy - using the transfer matrix - leading to the same quantum Hamiltonian, but in a more direct way.

The classically improved quantum Hamiltonian obtained in this way has the mathematical structure of a kinetic part with an infinite number of terms. The reason for this structure is given by the fact that the inversion of a nearly local matrix leads to a non-local matrix. This being an undesirable feature from the point of view of practical calculations, we show that it is possible to use the non-uniqueness of the improved action to obtain an improved quantum Hamiltonian containing only nearest neighbor interaction terms.

Finally we discuss the structure of the quantum Hamiltonian related to the tadpole and Lüscher-Weisz corrections. The determination of the corresponding coefficients, however, will be deferred to a future investigation. Although we do not discuss the third strategy in detail, a fully consistent computation of the quantum corrections should eventually lead to the same result as the first two strategies. Note that we discuss here only the improved Hamiltonian for the purpose to compute the spectrum. The general existence of such improved Hamiltonians is discussed in standard many-body theory in the context of model space calculations ([15, 16]). Like in the action formulation general observables require particular improved operators.

Calculations of the glueball spectrum using the coupled cluster method based on the standard Kogut-Susskind Hamiltonian have been done by Luo et al. [8] and Schütte et al. [9]. An incorporation of an improved Hamiltonian should be possible and one would expect reliable results already in lower order of the coupled cluster truncation compared to the standard Kogut-Susskind Hamiltonian. Calculations in this direction are under way[10].

2 From Wilson action to Kogut-Susskind Hamiltonian

2.1 Canonical method via Legendre transformation

Before deriving the improved Hamiltonian, we describe in a pedagogical manner how to obtain the standard Kogut-Susskind lattice Hamiltonian [4] from the classical lattice Lagrangian using the Legendre transformation [5, 6, 7], and canonical quantization. Wilson's Euclidean lattice action is given by ($a \equiv a_s$)

$$S_E = \frac{a}{a_t} \frac{2N_c}{g^2} \sum_{t-\square} (1 - P_\square) + \frac{a_t}{a} \frac{2N_c}{g^2} \sum_{s-\square} (1 - P_\square). \quad (1)$$

Here the notation of Ref.[13] has been used, i.e., $t - \square$ stands for time-like and $s - \square$ for space-like plaquettes, respectively, and

$$P_\square = \frac{1}{N_c} \text{Re } \text{Tr}(U_\square). \quad (2)$$

For later use we need to distinguish between the Euclidean and Minkowski action as well as Lagrangian. Its relation is defined, when going from Minkowski to Euclidean time by the transformation $it \rightarrow t$, $\exp[iS_M] \rightarrow \exp[-S_E]$, $L_M \rightarrow -L_E$ and $S_M = \int dt L_M$ as well as $S_E = \int dt L_E$. Thus the Euclidean lattice Lagrangian is the following

$$\begin{aligned} L_E &= -\frac{2N_c a}{g^2 a_t^2} \sum_{x,i} (P_{io} - 1) - \frac{2N_c}{g^2 a} \sum_{x,i < j} (P_{ij} - 1) \\ &= -\frac{a}{g^2 a_t^2} \sum_{x,i} \text{Tr}(U_{i0}(x) + U_{i0}^\dagger(x) - 2) \\ &\quad - \frac{1}{g^2 a} \sum_{x,i < j} \text{Tr}(U_{ij}(x) + U_{ij}^\dagger(x) - 2). \end{aligned} \quad (3)$$

Here $U_{ij}(x, t)$ denotes a space-like plaquette where the first link goes in direction i and the second link goes in direction j , and $U_{i0}(x, t)$ denotes the corresponding time-like plaquette. One should note that $P_{i0} = P_{0i}$ and $P_{ij} = P_{ji}$ for plaquettes. In the temporal gauge $U_0(x, t) = 1$, the time-like plaquette becomes a function only of the link variables $U_i(x, t)$ ($i = 1, 2, 3$) and we have

$$P_{i0} = \frac{1}{N_c} \text{Re Tr}(U_{i0}) = \frac{1}{N_c} \text{Re Tr} \left(U_i(x, t) U_i^\dagger(x, t + a_t) \right). \quad (4)$$

We want to construct a classical Lagrangian defining trajectories of generalized coordinates $U_i(x, t)$ and generalized velocities, where the variable t is now continuous. We assume that the action corresponding to this Lagrangian is given by the continuum limit $a_t \rightarrow 0$ of the lattice action which also yields the dependence on the generalized velocities.

In order to construct this Lagrangian, we introduce a Taylor expansion in time and write for a fixed (i, x) up to errors of $O(a_t^3)$

$$\begin{aligned} P_{i0} - 1 &= \frac{1}{N_c} \text{Re Tr}[U(t)U^\dagger(t + a_t) - 1] \\ &= \frac{1}{N_c} \text{Re Tr} \left(U(t)[U^\dagger(t) + a_t \dot{U}^\dagger(t) + \frac{a_t^2}{2} \ddot{U}^\dagger(t)] - 1 \right) \\ &= \frac{a_t^2}{2N_c} \text{Re Tr}[U(t)\ddot{U}^\dagger(t)] = -\frac{a_t^2}{2N_c} \text{Tr}[\dot{U}^\dagger(t)\dot{U}(t)] \\ &= -\frac{a_t^2}{2N_c} \text{Tr}[\dot{q}(t)\dot{q}(t)]. \end{aligned} \quad (5)$$

Here, we denote $U(t) \equiv U_i(x) \equiv U_i(x, t)$, and we have introduced a generalized velocity \dot{q} (corresponding to the angular velocities of the classical top theory) which is an element of the $SU(N_c)$ Lie algebra,

$$\dot{q}_i(x) = \dot{q}^\alpha(x) \lambda^\alpha = -i \dot{U}_i(x) U_i^\dagger(x) = i U_i(x) \dot{U}_i^\dagger(x). \quad (6)$$

The $SU(N_c)$ generators λ^a are normalized to $\text{tr}(\lambda^a \lambda^b) = \delta^{ab}/2$. Going to the limit $a_t \rightarrow 0$ and performing a transition from Euclidean to Minkowski space ($t \rightarrow it$) yields the classical lattice Lagrangian

$$L_M = \frac{a}{2g^2} \sum_{x,i} \dot{q}_i^\alpha(x) \dot{q}_i^\alpha(x) + \frac{1}{g^2 a} \sum_{x,i < j} \text{Tr}(U_{ij} + U_{ij}^\dagger - 2). \quad (7)$$

Here we denote $U_{ij} \equiv U_{ij}(x) \equiv U_{ij}(x, t)$. For a classical canonical formulation we introduce the conjugate momenta

$$\begin{aligned} E_j^\alpha(x) &= \frac{\partial L_M}{\partial \dot{q}_j^\alpha(x)} = \frac{a}{g^2} \dot{q}_j^\alpha(x) = \frac{2ai}{g^2} \text{Tr}[\lambda^\alpha U_j(x) \dot{U}_j^\dagger(x)], \\ E_j(x) &= E_j^\alpha(x) \lambda^\alpha = \frac{a}{g^2} \dot{q}_j(x). \end{aligned} \quad (8)$$

The standard Legendre transformation leads then to the following classical lattice Hamiltonian

$$\begin{aligned} H &= \sum_{x,i} \frac{\partial L_M}{\partial \dot{q}_i^\alpha(x)} \dot{q}_i^\alpha(x) - L_M \\ &= \frac{g^2}{2a} \sum_{x,i} E_i^\alpha(x) E_i^\alpha(x) - \frac{1}{g^2 a} \sum_{x,i < j} \text{Tr}(U_{ij} + U_{ij}^\dagger - 2). \end{aligned} \quad (9)$$

Recalling $\dot{q}_i \rightarrow ga\dot{A}$, we convince ourselves that $E_i \approx a^2 \dot{A}_i/g$. Therefore $E_i(x)$ is the approximated color-electric field on the lattice.

To quantize this classical theory, we proceed according to the rules of quantization of the classical top theory[14]. This results in the prescription that the quantum mechanical states are functions of the link variables $U_i(x)$ and that the canonical conjugates $E_i(x)$ become the operators of infinitesimal left multiplication. A generalization of the standard quantum mechanical formula $x + a = e^{-i\hat{p}a} x e^{i\hat{p}a}$ yields for the link variables

$$e^{i\epsilon^\alpha \lambda^\alpha} U_j(x) = e^{-i\epsilon^\alpha E_j^\alpha(x)} U_j(x) e^{i\epsilon^\alpha E_j^\alpha(x)}. \quad (10)$$

Variables corresponding to different lattice links are considered to be independent. This yields the commutation relations

$$\begin{aligned} [U_i(x), E_j^\alpha(y)] &= \lambda^\alpha U_i(x) \delta_{x,y} \delta_{i,j}, \\ [U_i^\dagger(x), E_j^\alpha(y)] &= -U_i^\dagger(x) \lambda^\alpha \delta_{x,y} \delta_{i,j}. \end{aligned} \quad (11)$$

Since the operators $e^{iE_j^\alpha(x)\lambda^\alpha}$ yield a representation of the gauge group $SU(N_c)$, we obtain for the $E_j(x)$ the commutation relations of the Lie algebra

$$[E_j^\alpha(x), E_j^\beta(x)] = if^{\alpha\beta\gamma} E_j^\gamma(x). \quad (12)$$

The quantization of the classical Hamiltonian, Eq.(9), by use of the commutation relations Eq.(11) gives the standard quantum Hamiltonian of Kogut and Susskind.

2.2 The transfer matrix method

The construction of the Kogut Susskind Hamiltonian from the Wilson action via the transfer matrix method has been first established by Creutz[13] (see also Ref.[11]). Here we recall the basic steps which may be used also for the construction of the improved Hamiltonian as discussed below.

$$S_E = \int dt L_E = \sum_t a_t [L_0(q(t), q(t + a_t)) + L_1(q(t))] + O(a_t^2). \quad (13)$$

Hereby, L_0 is kinetic part of the Lagrangian which couples the system at time t to that at time $t + a_t$. Invoking the Baker-Cambell-Hausdorf formula and going to the limit $a_t \rightarrow 0$ [12], the Hamiltonian is eventually given by

$$H = H_0 + L_1, \quad (14)$$

where the nontrivial part H_0 is related to L_0 via the functional integral kernel of the corresponding time evolution operator (transfer matrix formalism).

Let us recall that relation for the simple example of standard one-body quantum mechanics of free motion [12] where one has (we put the mass $m = 1$)

$$L_0(q', q) = \frac{1}{2a_t^2} (q' - q)^2, \quad (15)$$

with $q' = q(t + a_t)$, $q = q(t)$. The discrete time-evolution, which relates the generator H_0 to the kernel L_0 , is given by

$$(e^{-a_t H_0} \psi)(q) = N \int dq' e^{-a_t L_0(q, q')} \psi(q'), \quad (16)$$

where N is some unimportant normalization factor. Using

$$\psi(q') = e^{(q' - q)\nabla} \psi(q), \quad (17)$$

yields

$$e^{-a_t H_0} = N \int dq' e^{-a_t L_0(q, q')} e^{(q' - q)\nabla}. \quad (18)$$

In this case, and also for the case of scalar field theory [12], this integral is analytically computable for finite a_t . It yields the usual result $H_0 = -\Delta/2$.

In order to clarify the notations and the particularities for non-abelian gauge field theories, we first recall how to obtain from the transfer matrix method the standard Kogut Susskind Hamiltonian. We start by the decomposition of the action as given by Eq.(1) and introduce the temporal gauge. The corresponding Lagrangian is given by

$$\begin{aligned} L_0(U(t + a_t), U(t)) &= \frac{a}{g^2 a_t^2} \sum_{x,i} \text{Tr}(2 - U_i(x, t + a_t) U_i^\dagger(x, t) - h.c.), \\ L_1(U(t)) &= \frac{1}{g^2 a} \sum_{x,i < j} \text{Tr}(2 - U_{ij}(x, t) - h.c.). \end{aligned} \quad (19)$$

Here L_0 corresponds to the kinetic part and L_1 to the potential part, respectively. The kinetic part of the Lagrangian is given by the plaquettes involving different times. Using the notation analogous to Eq.(15),

$$\begin{aligned} L_0(U'_{x,j}, U_{x,j}) &= \frac{a}{g^2 a_t^2} \sum_{x,j} \text{Tr} \left[2 - (V_{x,j} + V_{x,j}^\dagger) \right], \\ V_{x,j} &= U'_{x,j} U_{x,j}^\dagger, \end{aligned} \quad (20)$$

where U corresponds to the time slice t and U' to the time slice $t + a_t$, respectively. It is well known that the quantum mechanics of $SU(N_c)$ gauge theory and that of the quantum mechanical top are closely related [4, 14]. Thus the relation between the wave function at time slice t and time slice $t + a_t$, in analogy to Eq.(17), involves the standard color electric operators E^α , $\alpha = 1, \dots, N_c^2 - 1$,

$$\Psi(U') = e^{i\omega^\alpha E^\alpha} \Psi(U). \quad (21)$$

where the numbers ω^α are the parameters of a group element $g_V \in SU(N_c)$ such that

$$g_V = e^{i\omega^\alpha \lambda^\alpha} = U' U^{-1} = V. \quad (22)$$

The analogy to Eq.(18) the Hamiltonian H_0 in the case of one link variable $V = U' U^\dagger$ is given by

$$e^{-a_t H_0} = N \int dU' e^{-a_t L_0(V)} e^{i\omega^\alpha E^\alpha}, \quad (23)$$

where $L_0(V)$ is given by Eq.(20). We use the invariance of the Haar measure yielding $dU' = dV$ and note that like in Eqs.(17,21), the operators E^a have to

be treated as commuting with U and U' . The integral in Eq.(23) can not be evaluated analytically for finite time translations a_t . However, for the determination of H_0 one may consider the limit $a_t \rightarrow 0$. In this case, the variables V approach the identity and it is legitimate to use the approximation for the L_0 term

$$Tr(V + V^\dagger) = 2Tr(\cos \lambda^\alpha \omega^\alpha) = 2 \left[N_c - \frac{1}{4} \omega^2 + O(\omega^4) \right]. \quad (24)$$

Writing the group integral $\int dU$ as Haar measure $\int \prod_a d\omega^a det_{Jac}$ yields for Eq.(23) a Gaussian integral in analogy to Eq.(18), with the exponent

$$\frac{a}{2a_t g^2} \omega^\alpha \omega^\alpha + i E^\alpha \omega^\alpha = \frac{a}{2a_t g^2} (\omega^\alpha + i E^\alpha \frac{a_t g^2}{a})^2 + \frac{a_t g^2}{2a} E^\alpha E^\alpha. \quad (25)$$

Taking the sum over the space-like links this reproduces the standard kinetic term of the Kogut Susskind Hamiltonian

$$H_0 = \frac{g^2}{2a} \sum_{x,j} E_j^\alpha(x) E_j^\alpha(x). \quad (26)$$

3 Classical improvement of Kogut Susskind Hamiltonian

3.1 Continuum behavior of classical improved action

The Wilson action reproduces the classical continuum action only up to errors of $O(a^2)$. It is possible to add to the Wilson action new terms such that these $O(a^2)$ errors are canceled [17, 18, 1]. In order to construct the corresponding improved Hamiltonian, one needs a generalization to lattices with $a_t \neq a_s \equiv a$. We first discuss the classical continuum behavior of the Wilson action. For space-like plaquettes one has (see Ref.[1])

$$P_{ij} = \frac{1}{N_c} Re Tr(U_{ij}) \rightarrow 1 - \frac{g^2 a^4}{2N_c} \left(Tr[F_{ij} F_{ij}] + \frac{a^2}{12} Tr[F_{ij} (\mathcal{D}_i^2 + \mathcal{D}_j^2) F_{ij}] \right). \quad (27)$$

For convenience, the continuum limit of a loop is expressed in terms of the field strength tensor and its covariant derivative the center of the loop x_0 .

According to Ref.[1] for time-like plaquettes one has to consider the path ordered integral

$$\begin{aligned}\oint A \cdot dx &\rightarrow \int_{-a/2}^{a/2} dx_i \int_{-a_t/2}^{a_t/2} dt [F_{i0}(x_0) + \frac{1}{2} x_{i'} x_{j'} (\mathcal{D}_{i'} \mathcal{D}_{j'} F_{i0})|_{x=x_0}] \\ &\rightarrow aa_t F_{i0}(x_0) + \frac{a_t a^3}{24} (\mathcal{D}_i^2) F_{i0}(x_0) + \frac{aa_t^3}{24} (\mathcal{D}_0^2) F_{i0}(x_0).\end{aligned}\quad (28)$$

The last term can be neglected since $a_t \ll a$. Therefore

$$\begin{aligned}P_{i0} &= \frac{1}{N_c} \text{Re Tr}(U_{i0}) \rightarrow \frac{1}{N_c} \text{Re Tr}[1 - \frac{1}{2} (g \oint A \cdot dx)^2] \\ &\rightarrow \frac{1}{N_c} \text{Re Tr}[1 - \frac{g^2 a^2 a_t^2}{2} (F_{i0} + \frac{1}{24} a^2 \mathcal{D}_i^2 F_{i0})^2] \\ &\rightarrow 1 - \frac{g^2 a^2 a_t^2}{2 N_c} \left(\text{Tr}[F_{i0} F_{i0}] + \frac{a^2}{12} \text{Tr}[F_{i0} \mathcal{D}_i^2 F_{i0}] \right) + O(a^2 a_t^4).\end{aligned}\quad (29)$$

In order to compensate these $O(a^2)$ errors, Lepage[1] has proposed to add new terms to the Wilson action. One of these terms is given by a rectangular loop,

$$R_{\mu\nu} = \frac{1}{N_c} \text{Re Tr} \left[\begin{array}{c} \overleftarrow{\hspace{0.5cm}} \overrightarrow{\hspace{0.5cm}} \\ \overrightarrow{\hspace{0.5cm}} \overleftarrow{\hspace{0.5cm}} \end{array} \right] \begin{array}{c} \nu \\ \uparrow \\ \mu \end{array} \quad (30)$$

For a space-like loop one has in particular

$$\begin{aligned}R_{ij} &= \frac{1}{N_c} \text{Re Tr}[U_i(x) U_i(x + a\hat{i}) U_j(x + 2a\hat{i}) \\ &\quad \times U_i^\dagger(x + a\hat{j} + a\hat{i}) U_i^\dagger(x + a\hat{j}) U_j^\dagger(x)] \\ &\rightarrow 1 - \frac{g^2 a^4}{2 N_c} \left(4 \text{Tr}[F_{ij} F_{ij}] + \frac{a^2}{3} \text{Tr}[F_{ij} (4\mathcal{D}_i^2 + \mathcal{D}_j^2) F_{ij}] \right).\end{aligned}\quad (31)$$

Considering time-like loops, there are two possibilities. First, one has a $2a \times a_t$ rectangular loop

$$R_{i0} = \frac{1}{N_c} \text{Re Tr} \left[\begin{array}{c} \overleftarrow{\hspace{0.5cm}} \overrightarrow{\hspace{0.5cm}} \\ \overrightarrow{\hspace{0.5cm}} \overleftarrow{\hspace{0.5cm}} \end{array} \right] \begin{array}{c} 0 \\ \uparrow \\ i \end{array} \quad (32)$$

yielding

$$\begin{aligned}
R_{i0} &= \frac{1}{N_c} \text{Re Tr} [U_i(x, t) U_i(x + a\hat{i}, t) U_0(x + 2a\hat{i}, t) \\
&\times U_i^\dagger(x + a\hat{i}, t + a_t) U_i^\dagger(x, t + a_t) U_0^\dagger(x, t)] \\
&\rightarrow 1 - \frac{g^2 a^2 a_t^2}{2N_c} \text{Tr Re} (2F_{i0} + \frac{a^2}{3} \mathcal{D}_i^2 F_{i0})^2 \\
&\rightarrow 1 - \frac{g^2 a^2 a_t^2}{2N_c} \left(4\text{Tr} [F_{i0} F_{i0}] + \frac{4a^2}{3} \text{Tr} [F_{i0} \mathcal{D}_i^2 F_{i0}] \right). \quad (33)
\end{aligned}$$

Secondly, one has a $2a_t \times a$ rectangular loop

$$R_{0i} = \frac{1}{N_c} \text{Re Tr} \left[\begin{array}{c} \text{---} \text{---} \text{---} \\ \updownarrow \quad \updownarrow \\ \text{---} \text{---} \text{---} \end{array} \right] \quad \begin{array}{c} 0 \\ \downarrow \\ i \end{array} \quad (34)$$

This term corresponds to advancing two steps in time direction. The conventional transfer matrix corresponds to an advance of a single step in time direction. Thus it is not compatible with the definition of the transfer matrix. We may disregard this term because the improvement terms in the Lagrangian are not uniquely determined[1]. Taking into account only the first term is sufficient.

Therefore, we make the following ansatz for the classically improved Euclidean lattice Lagrangian [1].

$$\begin{aligned}
L_t &= -\frac{2N_c a}{g^2 a_t^2} \sum_{x,i} [C'_1 \frac{P_{i0} + P_{0i}}{2} + C'_2 R_{i0}] + \text{const.} \\
C'_1 &= 4/3 \quad \text{and} \quad C'_2 = -1/12, \\
L_s &= -\frac{2N_c}{g^2 a} \sum_{x,i < j} [C_1 \frac{P_{ij} + P_{ji}}{2} + C_2 (R_{ij} + R_{ji})] + \text{const.} \\
C_1 &= 5/3 \quad \text{and} \quad C_2 = -1/12, \\
L_E &= L_t + L_s. \quad (35)
\end{aligned}$$

3.2 Improved Hamiltonian via Legendre transformation

Now we proceed as in sect.(2.1) to construct a classical Lagrange function in Minkowski space in terms of the generalized coordinates $U_i(x)$ and the generalized velocities $\dot{q}_i(x)$ as defined in Eq.(6). Working in the temporal gauge and denoting $\bar{U}(t) = U_i(x + a\hat{i}, t)$ yields

$$\begin{aligned}
R_{i0} &= \frac{1}{N_c} \text{Re} \text{Tr} [\bar{U}(t) \bar{U}^\dagger(t + a_t) U^\dagger(t + a_t) U(t)] \\
&\rightarrow \frac{1}{N_c} \text{Re} \text{Tr} \{ \bar{U}(t) [\bar{U}^\dagger(t) + a_t \dot{\bar{U}}^\dagger(t) + \frac{a_t^2}{2} \ddot{\bar{U}}^\dagger(t)] \\
&\quad \times [U^\dagger(t) + a_t \dot{U}^\dagger(t) + \frac{a_t^2}{2} \ddot{U}^\dagger(t)] U(t) \} \\
&= \frac{1}{N_c} \text{Re} \text{Tr} [\frac{a_t^2}{2} \bar{U}(t) \ddot{\bar{U}}^\dagger(t) + \frac{a_t^2}{2} \ddot{U}^\dagger(t) U(t) + a_t^2 \bar{U}(t) \dot{\bar{U}}^\dagger(t) \dot{U}^\dagger(t) U(t)] \\
&\quad + \text{const.} \\
&\rightarrow -\frac{1}{N_c} a_t^2 \text{Tr} [\frac{1}{2} \dot{q}_i(x + a\hat{i}) \dot{q}_i(x + a\hat{i}) + \frac{1}{2} \dot{q}_i(x) \dot{q}_i(x) + \dot{Q}_i(x) \dot{q}_i(x + a\hat{i})] \\
&\quad + \text{const.}, \tag{36}
\end{aligned}$$

where we have introduced the variable

$$\dot{Q}_i(x) = U_i(x)^\dagger \dot{q}_i(x) U_i(x). \tag{37}$$

This gives the following classical improved lattice Lagrangian in Minkowski space

$$L_M = \frac{a}{g^2} \sum_{x,i} \text{Tr} [(C'_1 + 2C'_2) \dot{q}_i(x) \dot{q}_i(x) + 2C'_2 \dot{Q}_i(x) \dot{q}_i(x + a\hat{i})] - L_s. \tag{38}$$

This Lagrangian can be written in the form

$$\begin{aligned}
L_M &= \frac{1}{2} \frac{a}{g^2} \sum_{\sigma, \rho} \dot{q}_\sigma(t) M_{\sigma\rho}(U(t)) \dot{q}_\rho(t) - L_s, \\
&\quad \text{where } \sigma = (x, i, \alpha), \quad \rho = (y, j, \beta),
\end{aligned}$$

$$\begin{aligned}
M_{\sigma, \rho}(U(t)) &= (C'_1 + 2C'_2) \delta_{\sigma, \rho} \\
&\quad + 4C'_2 \delta(x, y - a\hat{i}) \delta_{i, j} \text{Tr} [U_i^\dagger(x) \lambda^\alpha U_i(x) \lambda^\beta]. \tag{39}
\end{aligned}$$

The matrix M is not symmetric. However, it can be shown that only the symmetric part of M will contribute to the Hamiltonian. Thus we introduce

$$\begin{aligned} L_{\sigma,\rho}^{sym} &= \delta(x, y - a\hat{i})\delta_{i,j}Tr[U_{x\rightarrow y}^\dagger \lambda^\alpha U_{x\rightarrow y} \lambda^\beta] \\ &+ \delta(x, y + a\hat{i})\delta_{i,j}Tr[U_{y\rightarrow x}^\dagger \lambda^\beta U_{y\rightarrow x} \lambda^\alpha], \end{aligned} \quad (40)$$

which allows to write

$$M_{\sigma,\rho}^{sym} = \frac{1}{2}(M + M^t)_{\sigma,\rho} = (C'_1 + 2C'_2)\delta_{\sigma,\rho} + 2C'_2 L_{\sigma,\rho}^{sym}. \quad (41)$$

Inspection shows that L_{sym} and hence M_{sym} are real, symmetric matrices. Then the Lagrangian reads

$$L_M = \frac{1}{2} \frac{a}{g^2} \sum_{\sigma,\rho} \dot{q}_\sigma(t) M_{\sigma\rho}^{sym}(U(t)) \dot{q}_\rho(t) - L_s. \quad (42)$$

Via Legendre transformation, the classical improved Hamiltonian is obtained

$$\begin{aligned} H &= \sum_{x,i} \frac{\partial L_M}{\partial \dot{q}_i^\alpha(x)} \dot{q}_i^\alpha(x) - L_M = H_0 + V, \\ H_0 &= \frac{1}{2} \frac{g^2}{a} \sum_{\sigma,\rho} E_\sigma (M_{\sigma,\rho}^{sym})^{-1} E_\rho, \\ V &= -\frac{2N_c}{g^2 a} \sum_{x,i < j} \left[C_1 \frac{P_{ij} + P_{ji}}{2} + C_2 (R_{ij} + R_{ji}) \right]. \end{aligned} \quad (43)$$

The color-electric field E_σ is given by the conjugate momentum, being related to the generalized velocity \dot{q}_σ via,

$$E_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = \frac{a}{g^2} \sum_\rho M_{\sigma,\rho}^{sym} \dot{q}_\rho. \quad (44)$$

The color-electric field, obeys commutation relations with the link variables given by Eq.(11).

3.2.1 Hopping expansion and algebraic properties of M_{sym}

Taking a closer look to the kinetic part of the improved Hamiltonian reveals that via M_{sym}^{-1} an infinite number of terms enters into the Hamiltonian. In

analogy to the hopping parameter expansion [11], which expresses the propagator in terms of powers of a hopping matrix, we introduce K^{sym}

$$\begin{aligned} M_{sym} &= m_0 [1 + K_{sym}] = m_0 [1 + k_0 L_{sym}] \\ m_0 &= C'_1 + 2C'_2 = \frac{7}{6}, \\ k_0 &= \frac{2C'_2}{C'_1 + 2C'_2} = -\frac{1}{7}, \end{aligned} \quad (45)$$

to obtain

$$M_{sym}^{-1} = \frac{1}{m_0} [1 - K_{sym} + K_{sym}^2 - K_{sym}^3 + \dots]. \quad (46)$$

While K_{sym} involves only link variables between next neighbor lattice sites, higher powers of K_{sym} involve links extending over several lattice sites. Using the notation

$$U_{x \rightarrow x+(N+1)a\hat{i}} = U_i(x)U_i(x+a\hat{i}) \cdots U_i(x+Na\hat{i}), \quad (47)$$

we generalize the definition of L_{sym} to

$$\begin{aligned} L_{\sigma,\rho}^{(N)sym} &= \delta(x, y - Na\hat{i})\delta_{i,j} Tr[U_{x \rightarrow y}^\dagger \lambda^\alpha U_{x \rightarrow y} \lambda^\beta] \\ &+ \delta(x, y + Na\hat{i})\delta_{i,j} Tr[U_{y \rightarrow x}^\dagger \lambda^\beta U_{y \rightarrow x} \lambda^\alpha], \\ N &= 0, 1, 2, \dots, \end{aligned} \quad (48)$$

where $L_{sym}^{(0)} = 1$, and $L_{sym}^{(1)} = L_{sym}$. A little algebra shows that the matrix $L_{sym}^{(N)}$ obeys the following product rule

$$L_{sym}^{(p)} L_{sym}^{(q)} = \frac{1}{2} L_{sym}^{(p+q)} + \frac{1}{2} L_{sym}^{(|p-q|)}. \quad (49)$$

Thus we obtain for the lowest powers of K_{sym}

$$\begin{aligned} K_{sym} &= k_0 L_{sym}^{(1)}, \\ K_{sym}^2 &= k_0^2 \left[\frac{1}{2} L_{sym}^{(0)} + \frac{1}{2} L_{sym}^{(2)} \right], \\ K_{sym}^3 &= k_0^3 \left[\left(\frac{1}{2} + \frac{1}{4} \right) L_{sym}^{(1)} + \frac{1}{4} L_{sym}^{(3)} \right], \\ K_{sym}^4 &= k_0^4 \left[\left(\frac{1}{4} + \frac{1}{8} \right) L_{sym}^{(0)} + \frac{1}{2} L_{sym}^{(2)} + \frac{1}{8} L_{sym}^{(4)} \right], \end{aligned}$$

$$\begin{aligned}
K_{sym}^5 &= k_0^5 \left[\left(\frac{1}{2} + \frac{1}{8}\right) L_{sym}^{(1)} + \left(\frac{1}{4} + \frac{1}{16}\right) L_{sym}^{(3)} + \frac{1}{16} L_{sym}^{(5)} \right], \\
K_{sym}^6 &= k_0^6 \left[\left(\frac{1}{4} + \frac{1}{16}\right) L_{sym}^{(0)} + \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}\right) L_{sym}^{(2)} + \left(\frac{1}{8} + \frac{1}{16}\right) L_{sym}^{(4)} + \right. \\
&\quad \left. + \frac{1}{32} L_{sym}^{(6)} \right], \\
&\vdots
\end{aligned} \tag{50}$$

It has the general structure

$$K_{sym}^n = k_0^n \sum_{p=0}^n \kappa_p^{(n)} L_{sym}^{(p)}. \tag{51}$$

The coefficients of lowest order are

$$\begin{aligned}
\kappa_0^{(0)} &= 1, \\
\kappa_0^{(1)} &= 0, \quad \kappa_1^{(1)} = 1, \\
\kappa_0^{(2)} &= \frac{1}{2}, \quad \kappa_1^{(2)} = 0, \quad \kappa_2^{(2)} = \frac{1}{2}, \\
&\vdots
\end{aligned} \tag{52}$$

The coefficients $\kappa_p^{(n)}$ vanish except when n and p are both even or both odd. Using Eq.(51), we express M_{sym}^{-1} by

$$\begin{aligned}
M_{sym}^{-1} &= \frac{1}{m_0} \sum_{p=0}^{\infty} \mu_p L_{sym}^{(p)}, \\
\mu_p &= \sum_{n=p}^{\infty} (-k_0)^n \kappa_p^{(n)}.
\end{aligned} \tag{53}$$

As result, starting from an improved Lagrangian with a finite number of terms, one obtains for the improved Hamiltonian an expression given by an infinite number of terms.

In the following we will explore more of the algebraic structure of M_{sym} and obtain analytic expressions for the hopping expansion coefficients $\kappa_p^{(n)}$. This will be useful in what follows. We introduce

$$J_{\sigma,\rho} = 2\delta(x, y - a\hat{i}) \delta_{i,j} Tr[U_{x \rightarrow y}^\dagger \lambda^\alpha U_{x \rightarrow y} \lambda^\beta]. \tag{54}$$

A little algebra shows that

$$J_{\sigma,\rho}^n = 2\delta(x, y - na\hat{i})\delta_{i,j}Tr[U_{x\rightarrow y}^\dagger\lambda^\alpha U_{x\rightarrow y}\lambda^\beta], \quad (55)$$

and

$$JJ^t = J^tJ = 1, \quad (56)$$

i.e., J is a real, orthogonal matrix. Comparison with Eqs.(40, 48) shows

$$\begin{aligned} L_{sym} &= \frac{1}{2}(J + J^t), \\ L_{sym}^{(p)} &= \frac{1}{2}(J^p + (J^t)^p), \\ K_{sym} &= \frac{k_0}{2}(J + J^t), \\ M_{sym} &= m_0 \left[1 + \frac{k_0}{2}(J + J^t) \right]. \end{aligned} \quad (57)$$

Using Eq.(56), M_{sym} can be factorized,

$$M_{sym} = \frac{m_0}{1 + C^2}(1 + CJ)(1 + CJ^t), \quad (58)$$

if C is chosen as solution of

$$k_0 = \frac{2C}{1 + C^2}. \quad (59)$$

Solutions are $C = -7 \pm 4\sqrt{3}$. Note that J being a real, orthogonal matrix, which has eigenvalues of modulus one, and $|C| \neq 1$, thus the matrix M_{sym} can be inverted and M_{sym}^{-1} is well defined. Moreover, we note that M_{sym} is a positive definite matrix. This can be seen directly from Eq.(58), which factorizes M_{sym} into a matrix times its Hermitian conjugate. Also, a lower bound can be estimated using Eq.(57). J being orthogonal implies $||J|| = 1$. Thus R_J defined by $R_J = \frac{1}{2}(J + J^t)$, being a real, symmetric matrix like M_{sym} , obeys $||R_J|| \leq 1$. Then an arbitrary state vector ϕ of unit norm yields $|\langle \phi | R_J | \phi \rangle| \leq 1$. Then Eq.(57) implies

$$\begin{aligned} \langle \phi | M_{sym} | \phi \rangle &= m_0 + m_0 k_0 \langle \phi | R_J | \phi \rangle \\ &= \frac{7}{6} - \frac{1}{6} \langle \phi | R_J | \phi \rangle \geq 1, \end{aligned} \quad (60)$$

showing also that M_{sym} is positive. To summarize the properties of M_{sym} , this is a real, symmetric, positive definite and non-singular matrix. This property is needed for the construction of the Hamiltonian via the transfer-matrix, in particular for doing the Gaussian integral.

Factorization of M_{sym} , via Eq.(58), allows to express the kinetic energy term H_0 of the Hamiltonian, Eq.(43), as follows

$$\begin{aligned}
H_0 &= \frac{g^2}{2a} \frac{1+C^2}{m_0} \sum_{\nu} \left[\sum_{\rho} (1+CJ)_{\nu\rho}^{-1} E_{\rho} \right]^2 \\
&= \frac{g^2}{a} \frac{1+C^2}{m_0} Tr \sum_{x,i} [E_i(x) - CU_i(x)E_i(x+a\hat{i})U_i^{\dagger}(x) \\
&\quad + C^2U_i(x)U_i(x+a\hat{i})E_i(x+2a\hat{i})U_i^{\dagger}(x+a\hat{i})U_i^{\dagger}(x) - \dots]^2. \quad (61)
\end{aligned}$$

Note that this is an expansion in terms of C and J .

Analytic expressions for the coefficients of the hopping expansion can be obtained in the following way:

$$\begin{aligned}
K_{sym}^n &= \left(\frac{k_0}{2}\right)^n (J + J^t)^n \\
&= \left(\frac{k_0}{2}\right)^n \sum_{p=0}^n \binom{n}{p} J^p (J^t)^{n-p}. \quad (62)
\end{aligned}$$

Because this expression is a symmetric matrix and making use of Eqs.(56,57), one may write

$$K_{sym}^n = \left(\frac{k_0}{2}\right)^n \sum_{p=0}^n \alpha_p^{(n)} (J^p + (J^t)^p) = \left(\frac{k_0}{2}\right)^n \sum_{p=0}^n \alpha_p^{(n)} 2L_{sym}^{(p)}. \quad (63)$$

Comparison of coefficients yields

$$\begin{aligned}
p &= 0 : \alpha_0^{(n)} = \frac{1}{2} \binom{n}{n/2} \quad \text{if } n \text{ is even,} \\
&\quad \text{zero else,} \\
p &\geq 1 : \alpha_p^{(n)} = \binom{n}{(n+p)/2} \quad \text{if } n, p \text{ are both even or both odd,} \\
&\quad \text{zero else.} \quad (64)
\end{aligned}$$

Comparison with Eq.(51) eventually yields for the hopping expansion coefficients $\kappa_p^{(n)}$ the following expression,

$$\kappa_p^{(n)} = 2^{-n+1} \alpha_p^{(n)}. \quad (65)$$

3.3 Improved Hamiltonian via transfer matrix

We start from the classically improved Euclidean Lagrangian, given by Eq.(35). It is built from space-like plaquettes P_{ij} , time-like plaquettes P_{i0} and corresponding rectangular loops R_{ij} and R_{i0} . We now want to show that the transfer matrix method yields the same Hamiltonian as has been obtained in the previous section via Legendre transformation. Let us consider the time-like part of the Lagrangian, which yields the kinetic part of the Hamiltonian. The space-like part yields the potential part in a trivial way. Using the temporal gauge, one has

$$\begin{aligned} P_{i0} &= \frac{1}{2N_c} \text{Tr} \left[U_i(x, t) U_i^\dagger(x, t + a_t) + U_i(x, t + a_t) U_i^\dagger(x, t) \right] \\ &= \frac{1}{2N_c} \text{Tr} \left[V_i(x, t) + V_i^\dagger(x, t) \right], \end{aligned} \quad (66)$$

using the notation $V_i(x, t) = U_i(x, t + a_t) U_i^\dagger(x, t)$. Similarly one obtains for the rectangular loop

$$\begin{aligned} R_{i0} &= \frac{1}{2N_c} \text{Tr} \left[V_i(x, t) U_i(x, t) V_i(x + \hat{a}_i, t) U_i^\dagger(x, t) \right. \\ &\quad \left. + U_i(x, t) V_i^\dagger(x + \hat{a}_i, t) U_i^\dagger(x, t) V_i^\dagger(x, t) \right]. \end{aligned} \quad (67)$$

The Hamiltonian is defined via the transfer matrix like in Eq.(23). Because we consider $a_t \rightarrow 0$, the group integral will be dominated by group elements of $SU(N_c)$ in the neighborhood of the unit element. Thus one can expand the group elements $V_i(x, t)$ in a Taylor series of the Lie group parameters $\omega_{x,i}^\alpha(t)$,

$$V_i(x, t) = \exp[i\omega_{x,i}(t)] = 1 + i\omega_{x,i}(t) - \frac{1}{2}\omega_{x,i}(t)^2 + O(\omega^3), \quad (68)$$

where we denote $\omega_{x,i}(t) = \sum_\alpha \omega_{x,i}^\alpha(t) \lambda^\alpha$. Thus we arrive at

$$\text{Tr}[V_i(x, t) + V_i^\dagger(x, t)] = 2N_c - \frac{1}{2} \sum_\alpha \omega_{x,i}^\alpha(t)^2 + O(\omega^3), \quad (69)$$

and hence in the notation of Eq.(39),

$$\sum_{x,i} Tr[V_i(x,t) + V_i^\dagger(x,t)] = -\frac{1}{2} \sum_{\sigma\rho} \omega_\sigma(t) \delta_{\sigma\rho} \omega_\rho(t) + O(\omega^3) + \text{const.} \quad (70)$$

Carrying out the corresponding steps for the rectangular term, one obtains

$$\begin{aligned} & \sum_{x,i} Tr \left[V_i(x,t) U_i(x,t) V_i(x + a\hat{i}), t) U_i^\dagger(x,t) \right. \\ & + \left. U_i(x,t) V_i^\dagger(x + a\hat{i}), t) U_i^\dagger(x,t) V_i^\dagger(x,t) \right] \\ & = - \sum_{\sigma\rho} \omega_\sigma(t) \left[\delta_{\sigma\rho} + \frac{1}{2} J_{\sigma\rho} \right] \omega_\rho(t) + O(\omega^3) + \text{const}, \end{aligned} \quad (71)$$

where J is given by Eq.(54). Inserting this into the time-like Lagrangian L_t yields eventually

$$L_t = \frac{1}{2} \frac{a}{a_t^2 g^2} \sum_{\sigma\rho} \omega_\sigma(t) \left[(C'_1 + 2C'_2) \delta_{\sigma\rho} + 4C'_2 J_{\sigma\rho}^{sym} \right] \omega_\rho(t). \quad (72)$$

One should note that the matrix J is not symmetric. However, to the Lagrangian only the symmetric part $J_{sym} = (J + J^t)/2$ contributes. Note further that $J_{sym} = L_{sym}$ and $M_{sym} = (C'_1 + 2C'_2)1 + 4C'_2 J_{sym}$, being real, symmetric matrices. Thus we arrive at

$$L_t = \frac{1}{2} \frac{a}{a_t^2 g^2} \sum_{\sigma\rho} \omega_\sigma(t) M_{\sigma\rho}^{sym} \omega_\rho(t). \quad (73)$$

The transfer matrix is then given by

$$\begin{aligned} & \exp[-a_t H_0 + O(a_t^2)] \\ & = \int [\prod_{x,i} dV_i(x)] \exp[-a_t L_t(V_i(x))] \exp[i \sum_{x,i,\alpha} \omega_{x,i}^\alpha E_i^\alpha(x)] \\ & = \int [\prod_\sigma d\omega_\sigma \det J_{ac}] \exp \left[-\frac{1}{2} \frac{a}{a_t g^2} \sum_{\sigma\rho} \omega_\sigma M_{\sigma\rho}^{sym} \omega_\rho + i \sum_\sigma \omega_\sigma E_\sigma \right] \\ & = N \exp \left[-\frac{1}{2} \sum_{\sigma\rho} E_\sigma \left(\frac{a}{a_t g^2} M_{\sigma\rho}^{sym} \right)^{-1} E_\rho \right]. \end{aligned} \quad (74)$$

Thus we obtain

$$H_0 = \frac{g^2}{2a} \sum_{\sigma\rho} E_\sigma (M_{sym}^{-1})_{\sigma\rho} E_\rho. \quad (75)$$

in agreement with the result, Eq.(43), obtained via Legendre transformation.

4 Improved Hamiltonian given by finite number of terms

As was shown in the previous section, the kinetic energy of the classical improved Hamiltonian obtained directly from Lepage's action is given by an infinitive series of terms. Even though the series is rapidly convergent, such an Hamiltonian is too complicated for a practical calculation. Recalling that the purpose of classical improvement is to push the $O(a^2)$ error to order $O(a^4)$, we show here how to construct a simpler improved Hamiltonian corresponding to a finite number of terms to achieve such a goal. In the previous section we have seen that the infinite number of terms in the Hamiltonian arises due to the inversion of the matrix M_{sym} , which itself has only a finite number of terms. Thus it is plausible that in order to obtain a Hamiltonian given by a finite number of terms, one needs to start from a Lagrangian corresponding to a matrix M_{sym} with an infinite number of terms. Such a construction is possible, because the Lagrangian leading to improvement is not unique. We start by considering the following type of Wilson loop, which emerges as a generalization of the $2a \times a_t$ loop R_{i0} to a $(n+1)a \times a_t$ loop parallel transporter $R_{ni,0}$ given by

$$R_{ni,0} = \frac{1}{N_c} Re Tr \quad \begin{array}{c} \boxed{} \\ \end{array} \quad (76)$$

$x \qquad \qquad x_0 \qquad \qquad x + na\hat{i}$

In the temporal gauge it corresponds to the expression

$$\begin{aligned} R_{ni,0} &= \frac{1}{N_c} Re Tr [U_i(x) U_i(x + a\hat{i}) \cdots U_i(x + (n-1)a\hat{i}) U_i(x + na\hat{i}) \\ &\times U_i^\dagger(x + na\hat{i}, t + a_t) U_i^\dagger(x + (n-1)a\hat{i}) \cdots U_i^\dagger(x + a\hat{i}) U_i^\dagger(x, t + a_t)]. \end{aligned} \quad (77)$$

Note that for $n = 1$, $R_{ni,0}$ coincides with R_{i0} . The path-ordered integral of such a Wilson loop is given by

$$\oint A \cdot dx \rightarrow \int dx_i dt [F_{i0}(x_0) + \frac{1}{2} x_{i'} x_{j'} (\mathcal{D}_{i'} \mathcal{D}_{j'} F_{i0})|_{x=x_0}]$$

$$\begin{aligned}
&= 2aa_t F_{i0}(x_0) + \frac{1}{2}a_t \mathcal{D}_i^2 F_{i0}(x_0) \left[\int_{(n-1)a/2}^{(n+1)a/2} x^2 dx + \int_{-(n+1)a/2}^{-(n-1)a/2} x^2 dx \right] \\
&= 2aa_t F_{i0}(x_0) + \frac{3n^2 + 1}{12} a_t a^3 \mathcal{D}_i^2 F_{i0}(x_0).
\end{aligned} \tag{78}$$

Therefore, we obtain the following continuum behavior for the above Wilson loop

$$\begin{aligned}
R_{ni,0} &\rightarrow \frac{1}{N_c} \text{Re} \text{Tr} \left[1 - \frac{1}{2} \left(\oint A \cdot dx \right)^2 \right] \\
&\rightarrow 1 - \frac{g^2 a^2 a_t^2}{2N_c} \left(4\text{Tr}[F_{i0}F_{i0}] + (n^2 + 1/3)a^2 \text{Tr}[F_{i0}\mathcal{D}_i^2 F_{i0}] \right).
\end{aligned} \tag{79}$$

One verifies for $n = 1$ that Eq.(79) coincides with Eq.(33), as should be.

We make the following ansatz for the Euclidean lattice Lagrangian

$$L_t = -\frac{2N_c a}{g^2 a_t^2} A' \left[B' \sum_{x,i} P_{io}(x) + \sum_{n=1}^{\infty} C'^n \sum_{x,i} R_{ni,0}(x) \right]. \tag{80}$$

In order that the usual continuum limit of the Lagrangian is obtained and the $O(a^2)$ error is canceled, we imply from the continuum behavior of P_{i0} , Eq.(29), and of $R_{ni,0}$, Eq.(79), that the following conditions hold,

$$A' \left[B' + 4 \sum_{n=1}^{\infty} C'^n \right] = 1, \tag{81}$$

$$A' \left[\frac{B'}{12} + \sum_{n=1}^{\infty} \left(n^2 + \frac{1}{3} \right) C'^n \right] = 0. \tag{82}$$

We have deliberately introduced the coefficient B' . Choosing

$$B' = 1 - 2 \sum_{n=1}^{\infty} C'^n, \tag{83}$$

results in a simple expression of the Lagrangian expressed in terms of generalized coordinates and velocities. Using

$$\sum_{n=1}^{\infty} n^2 C'^n = C' \frac{\partial}{\partial C'} \left(C' \frac{\partial}{\partial C'} \right) \sum_{n=1}^{\infty} C'^n = \frac{C'(1+C')}{(1-C')^3}, \tag{84}$$

we obtain

$$\begin{aligned} A' &= \frac{1 - C'}{1 + C'}, \\ B' &= \frac{1 - 3C'}{1 - C'}, \end{aligned} \quad (85)$$

and C' is a root of

$$C'^3 + 11C'^2 + 11C' + 1 = 0. \quad (86)$$

This equation has three real roots, given by

$$\begin{aligned} C' &= -1, \\ C' &= -5 \pm 2\sqrt{6}. \end{aligned} \quad (87)$$

The root closest to zero is $C'_0 = -5 + 2\sqrt{6} = -0.101021 \dots$. In order to obtain the kinetic energy, we express $R_{ni,0}$ in terms of generalized coordinates and velocities,

$$\begin{aligned} R_{ni,0} &\rightarrow \frac{1}{N_c} \text{Re Tr} \left[[U_i^\dagger(x) + a_t \dot{U}_i^\dagger(x) + \frac{1}{2} a_t^2 \ddot{U}_i^\dagger(x)] U_i(x) U_i(x + a\hat{i}) \dots \right. \\ &\times U_i(x + (n-1)a\hat{i}) U_i(x + na\hat{i}) [U_i^\dagger(x + na\hat{i}) + a_t \dot{U}_i^\dagger(x + na\hat{i}) \\ &+ \frac{1}{2} a_t^2 \ddot{U}_i^\dagger(x + na\hat{i})] U_i^\dagger(x + (n-1)a\hat{i}) \dots U_i^\dagger(x + a\hat{i}) \Big] \\ &\rightarrow \frac{1}{N_c} \text{Tr} \left[1 - \frac{1}{2} a_t^2 \dot{q}_i(x) \dot{q}_i(x) - \frac{1}{2} a_t^2 \dot{q}_i(x + na\hat{i}) \dot{q}_i(x + na\hat{i}) \right. \\ &- \left. a_t^2 \dot{Q}_i(x + (n-1)a\hat{i}) \dot{q}_i(x + na\hat{i}) \right], \end{aligned} \quad (88)$$

where we have introduced

$$\begin{aligned} \dot{Q}_i(x + (n-1)a\hat{i}) &= U_i^\dagger(x + (n-1)a\hat{i}) \dots U_i^\dagger(x + a\hat{i}) U_i^\dagger(x) \dot{q}_i(x) U_i(x) \\ &\times U_i(x + a\hat{i}) \dots U_i(x + (n-1)a\hat{i}). \end{aligned} \quad (89)$$

Note again, for $n = 1$, $\dot{Q}_i(x + (n-1)a\hat{i})$ coincides with $\dot{Q}_i(x)$ defined in Eq. (37). Thus we can write the time-like part of the Minkowski lattice Lagrangian,

$$L_t^M = \frac{a}{g^2} A' \sum_{x,i} Tr \left[\dot{q}_i(x) \dot{q}_i(x) + 2 \sum_{n=1}^{\infty} C'^n \dot{Q}_i(x + (n-1)a\hat{i}) \dot{q}_i(x + na\hat{i}) \right]. \quad (90)$$

To find the kinetic energy of the improved Hamiltonian, it is convenient to express this in terms of the matrix J , defined by Eq.(54),

$$L_t^M = \frac{a}{2g^2} \sum_{\sigma\rho} \dot{q}_\sigma M'_{\sigma\rho} \dot{q}_\rho, \quad (91)$$

where

$$M' = A' \left[1 + 2 \sum_{n=1}^{\infty} C'^n J^n \right] = A' \frac{1 + C' J}{1 - C' J}. \quad (92)$$

The color-electric field is expressed as

$$E_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = \frac{a}{g^2} \sum_{\rho} M'^{sym}_{\sigma,\rho} \dot{q}_\rho. \quad (93)$$

where

$$M'^{sym} = \frac{1}{2}(M' + M'^t) = A' \frac{1 - C'^2}{(1 - C' J)(1 - C' J^t)} = \frac{(1 - C')^2}{(1 + C'^2) - C'(J + J^t)}. \quad (94)$$

If we choose C' such that $|C'| \neq 1$, e.g. $C'_0 = -0.101021 \dots$, then M'^{sym} is a real, symmetric, positive and non-singular matrix. Finally, we obtain the corresponding kinetic energy of the improved Hamiltonian, given by

$$\begin{aligned} H_0 &= \frac{1}{2} \frac{g^2}{a} \sum_{\sigma,\rho} E_\sigma (M'^{sym})_{\sigma,\rho}^{-1} E_\rho \\ &= \frac{g^2}{2a} \sum_{\sigma,\rho} \left[\frac{1 + C'^2}{(1 - C')^2} E_\sigma \delta_{\sigma,\rho} E_\rho - \frac{C'}{(1 - C')^2} E_\sigma (J + J^t)_{\sigma,\rho} E_\rho \right] \\ &= \frac{g^2}{2a} \sum_{\sigma,\rho} \left[\frac{1 + C'^2}{(1 - C')^2} E_\sigma \delta_{\sigma,\rho} E_\rho - \frac{2C'}{(1 - C')^2} E_\sigma J_{\sigma,\rho} E_\rho \right] \\ &= \frac{g^2}{a} Tr \sum_{x,i} \left[\frac{1 + C'^2}{(1 - C')^2} E_i(x) E_i(x) - \frac{2C'}{(1 - C')^2} U_i(x)^\dagger E_i(x) U_i(x) E_i(x + a\hat{i}) \right]. \end{aligned} \quad (95)$$

It consists of only two terms, which makes it convenient for practical calculations.

5 Tadpole Improvement

In the preceding section, we derived a classically improved Hamiltonian for gluons with $O(a^2)$ corrections. A very important step of the improvement program is to take into account quantum corrections corresponding to tadpole diagrams. Without such improvement, only part of the $O(a^2)$ errors are canceled. According to Lepage and Mackenzie, most of the tadpole contributions can be removed by dividing each link operator U_μ by the mean u_μ of the link. For asymmetric lattices, $a_t \ll a_s$, and small enough a_t we have $u_t = 1$ for time-like directions. In the Hamiltonian formulation, the mean u_s of a space-like link is defined by

$$u_s = \langle \Omega | P_{ij} | \Omega \rangle^{1/4}, \quad (96)$$

where $|\Omega\rangle$ is the vacuum of the improved Hamiltonian. Thus tadpole improvement of the lattice Lagrangian $L = L_t + L_s$, where L_t is given by Eq.(80) and L_s by Eq.(35), corresponds to the replacements

$$\begin{aligned} P_{ij} &\rightarrow P_{ij}/u_s^4, \\ R_{ij} &\rightarrow R_{ij}/u_s^6, \\ P_{i0} &\rightarrow P_{i0}/u_s^2, \\ R_{ni,0} &\rightarrow R_{ni,0}/u_s^{2n+2}. \end{aligned} \quad (97)$$

This is equivalent to the following replacement of constants

$$\begin{aligned} C_1 &\rightarrow C_1/u_s^4, \\ C_2 &\rightarrow C_2/u_s^6, \\ g_t &\rightarrow g_t u_s, \\ C' &\rightarrow C'/u_s^2, \end{aligned} \quad (98)$$

where we put $g = g_t$ in Eq.(80). For the transition to the Hamiltonian these redefinitions of the coefficients can be taken over yielding for the “two-term”

tadpole improved Hamiltonian ($C' = -0.101021$)

$$\begin{aligned}
H &= H_0 + V, \\
H_0 &= \frac{g_t^2 u_s^2}{a} \text{Tr} \sum_{x,i} \left[\frac{1 + C'^2/u_s^4}{(1 - C'/u_s^2)^2} E_i(x) E_i(x) \right. \\
&\quad \left. - \frac{2C'/u_s^2}{(1 - C'/u_s^2)^2} U_i(x)^\dagger E_i(x) U_i(x) E_i(x + a\hat{i}) \right], \\
V &= -\frac{2N_c}{g_s^2 a} \sum_{x,i < j} \left[\frac{C_1}{u_s^4} \frac{P_{ij} + P_{ji}}{2} + \frac{C_2}{u_s^6} (R_{ij} + R_{ji}) \right]. \tag{99}
\end{aligned}$$

Here, we have introduced different couplings in the kinetic and potential terms in order to allow for a “speed of light” correction as discussed in Ref.[23] (see below).

6 Further perturbative improvement

Tadpoles have been identified as an essential part of the problem when approaching the continuum limit of quantum field theory on the lattice. A systematic perturbative calculation on the lattice has been performed by Lüscher and Weisz[20]. This leads to the determination of additional terms in the Lagrangian needed to compensate errors. It turns out that such a further improved Lagrangian (for details see Ref.[19]) contains the same plaquettes and planar rectangle loop terms which occurred before, but with suitably redefined coefficients, plus a new term, being a non-planar ”parallelogram” loop, given by

$$C_{\mu\nu\sigma} = \frac{1}{N_c} \text{Re Tr} \quad \text{[Diagram: A non-planar parallelogram loop with arrows indicating a clockwise path around the loop.]} \quad . \tag{100}$$

It corresponds to

$$C_{x,\mu\nu\sigma} = \frac{1}{N_c} \text{Re Tr} (U_{x,\mu} U_{x+a\hat{\mu},\nu} U_{x+a\hat{\mu}+a\hat{\nu},\sigma} U_{x+a\hat{\nu}+a\hat{\sigma},\mu}^{-1} U_{x+a\hat{\sigma},\nu}^{-1} U_{x,\sigma}^{-1}). \tag{101}$$

The corresponding term occurring in the Lagrangian is proportional to

$$\sum_{x,\mu < \nu < \sigma} C_{x,\mu\nu\sigma}. \tag{102}$$

The structure of the corresponding improved Hamiltonian can be inferred from the improved Lagrangian as before: One introduces different lattice spacings $a_s = a$ and a_t and constructs the Hamiltonian by Legendre transformation and canonical quantization. Here, we refrain from discussing details and only give the general structure of emerging Hamiltonian.

(1) The *plaquette and planar rectangle loop* terms will give a part of the improved Hamiltonian which has the same form as before, only the weights of the individual terms will be different.

(2) The *space-like parallelogram loop* terms (i.e. $\mu\nu\sigma$ space-like) will yield a corresponding additional term in the potential part of the Hamiltonian.

(3) The *time-like parallelogram loop* terms (where either μ or ν or σ is time-like, the other two indices being space-like) produce a large number of different contributions to the Hamiltonian (with well defined weights). The final result for the improved Lagrangian has the structure

$$L = L_t(\dot{q}, U) + L_s(U), \quad (103)$$

with

$$L_t = \frac{a}{2g_t^2} \left[\sum_{\sigma\rho} \dot{q}_\sigma M_{\sigma\rho}(U) \dot{q}_\rho + \sum_\sigma A_\sigma(U) \dot{q}_\sigma + h.c. \right]. \quad (104)$$

A new feature is the occurrence of a term linear in \dot{q} . As before, $M(U)$ is a symmetric matrix of the form

$$M = 1 + \tilde{M}, \quad (105)$$

allowing the definition of M^{-1} by a geometric series expansion. Legendre transformation and quantization yields a Hamiltonian of the structure

$$H = H_0 + V, \quad (106)$$

with

$$H_0 = \frac{g_t^2}{2a} \sum_{\sigma\rho} \left[E_\sigma M_{\sigma\rho}^{-1} E_\rho - (A_\sigma M_{\sigma\rho}^{-1} E_\rho + h.c.) - 2A_\sigma M_{\sigma\rho}^{-1} A_\rho \right]. \quad (107)$$

7 Discussion

For the purpose of a numerical calculation, in particular for a comparison with lattice Monte Carlo results, the following points are important:

(1) As discussed in Refs.[22, 23], the scales related to the regularization of the gauge field theory in the Hamiltonian formulation as opposed to the Euclidean path integral formulation are different. This difference can be accounted for by introducing spacelike (g_s) and time-like couplings (g_t) which have a well defined relation to the "Lagrangian coupling" g . In one-loop approximation this relation is of the type

$$\frac{1}{g_\mu^2} = \frac{1}{g^2} + c_\mu, \quad (108)$$

where c_μ depends on the space-time dimension and on the type of the gauge group and is given in detail in Refs.[22, 24, 23].

(2) Because of this difference in the nature of the lattice regularization, all perturbative calculations which determine some non-classical improvement in the sense of Lüscher-Weisz have to be redone. Such a calculation can be done on an asymmetric Euclidean lattice with $a_t \ll a_s$ (see Ref.[23]).

(3) Tadpole improvement which has been considered by Lepage[19] in the Lagrangian framework corresponds in the Hamiltonian framework to an expression given by Eq.(99).

(4) A systematic determination of the Lüscher-Weisz improvement terms on asymmetric lattices in the Hamiltonian framework has still to be done. Since these additional corrections turn out to be small in the standard Euclidean framework (see Ref.[19]) - the most important correction coming from the inclusion of the tadpole terms - it should be worthwhile to work with the improved Hamiltonian given by Eq.(99), e.g., for the numerical simulation of glueballs.

To summarize, we have investigated in this paper two schemes of improvement of the Kogut Susskind Hamiltonian: If one starts from Lepage's Lagrangian, which is preferable for Monte Carlo simulations in the Lagrangian formulation, the corresponding Hamiltonian is given by an infinite series of terms which contain terms with arbitrary long range. In contrast, we have shown that by starting from a suitable Lagrangian with an infinite number of terms, one can get an improved Hamiltonian consisting of a finite small number of terms. This should be preferable for numerical computations in the Hamiltonian framework.

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